The role of degeneracy in the analogy between continuous variable and spin $1/2$ systems

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We point out limitations to the analogy between the continuous variable and spin 1/2 systems and show that the maximal violation of Bell inequality is related to an infinite degeneracy. We quantify non-maximal violation of the Bell-CHSH inequality and comment potential experimental implications of our work.

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I. Introduction

The generalization of Bell's inequalities to quantum systems with continuous variables (CV's) has been formulated in terms of the Wigner representation [1] or else using a map between the CV's and a spin 1/2 system [2]. In fact it has been shown [3] that the pseudospin operators introduced in [2] are a limiting case of the observable introduced by Gisin and Peres [4], establishing thus a bridge between the finite and infinite dimensional (continuous) systems. The relevance of such studies stem from the fact that the violation of Bell's inequalities are considered a measure of the nonlocality and quantum nature of the system under consideration. Conditions under which maximal violation is attained has also been discussed in detail [3].

The formulation of Bell inequalities [5] requires the existence of an observable with eigenvalues ± 1. Once the observable is chosen the Bell operator is built and the inequality is expressed in terms of the expectation value for a given state. The state used to define the expectation value is considered the source of entanglement and non-locality, and therefore of the inequality's violation. The authors in [2] considered the Bell inequality due to Clauser, Horne, Shimony and Holt [6] using the two mode squeezed vacuum state and the so called pseudospin operator, which are introduced to realize the mapping between the continuous variables and the spin 1/2. In this way, the authors show that the Bell-CHSH inequality can be maximally violated. It is worth remarking that the two mode squeezed vacuum state is a regularized version of the state used by EPR in his famous article [7]. In fact, the EPR state is obtained in the limit of infinite squeezing, coincidentally the maximal violation of the Bell CHSH is obtained also in that limit. On the other hand, it has been argued that the map between CV's and the pseudospin operator opens the possibility that the CV's systems may be exploited to do quantum information tasks which could be robust against photon losses.

This work is concerned with the generalization of the Bell's inequalities based upon the mapping between the CV's and a two qubit spin system. Two points motivated our study. First, one should be careful with such mapping since intuition tell us that the degrees of freedom, or the information contained in the CV's system should be reflected some how in the finite dimensional system (pseudospin). In fact, it turns out that the mapping necessarily lead to a strong degeneration. Second, it is known [8] that maximal violation is closely related to degeneration. The purpose of this paper is to quantify the role of degeneration in the violation of the inequalities, *i.e* to calculate the violation of the Bell-CHSH inequality as a function of the degeneracy. There are two ways to achieve our goal. The first, and more direct one, is based upon the use of different truncated versions (in the number representation basis) of the two mode squeezed state. The second approach, the one we follow here, use the concept of "entangled observable" [9]. The point is that usually the state used to define the expectation value is considered the source of violation of the inequality. However there exist also the possibility that are the observables - and not the states - which have the characteristic of being non-local and or entangled.

The possibility to transfer the entanglement from the state to the observable is a consequence of the fact that the two mode squeezed vacuum state can be obtained through an unitary (but non-local) transformation from the vacuum and that, as for any other unitary transformation, physics is the same in both cases. Assigning the entanglement to the states or to the observable are just two different ways to evaluate the matrix elements (although each procedure has its own interpretation), the advantage of the approach we use here is that it permits to gain insight into the role played by degeneracy. Section II is devoted to a brief revision of the conventional approach to the inequality, section III contains the contribution of this work and make a brief comment regarding the potential experimental implications of our work,

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finally in section IV we summarize our results.

II. Spin 1/2 analogy

The Bell-CHSH inequality for a two qubit system is expressed in terms of the Bell operator $\mathcal{B}$:

$$\mathcal{B} = (\mathbf{u}\cdot\hat{s}_1)\otimes(\mathbf{v}\cdot\hat{\mathbf{s}}_\mathbf{2}) + (\mathbf{u}\cdot\hat{s}_1)\otimes(\mathbf{v}'\cdot\hat{s}_2) + (\mathbf{u}'\cdot\hat{s}_1)\otimes(\mathbf{v}\cdot\hat{\mathbf{s}}_\mathbf{2}) - (\mathbf{u}'\cdot\hat{s}_1)\otimes(\mathbf{v}'\cdot\hat{s}_2). \quad (1)$$

where $\hat{s}_i$ is the Pauli matrix for the i-th qubit ($i = 1, 2$) and $\mathbf{u}, \mathbf{v}, \mathbf{u}'$ and $\mathbf{v}'$, are three dimensional unit vectors. For a local realistic theory the Bell-CHSH inequality holds [6] $|< \mathcal{B} >| \leq 2$ while for any two qubit state Cirel'son limit applies $|< \mathcal{B} >| \leq 2\sqrt{2}$.

Two mode squeezed vacuum states (TMSV) are relevant in the generalization of the Bell-CHSH inequality (1) for continuous variables (CV) systems since matrix elements are taken respect to these states. TMSV are constructed by means of the creation and annihilation operators $(a, a^\dagger)$ and $(b, b^\dagger)$ associated to the first and second channels respectively:

$$\begin{aligned} S(\zeta) &= e^{\zeta(a^\dagger b^\dagger - ab)} \\ |\zeta >= S(\zeta)|00> &= \sum_{n=0}^{\infty} \frac{(\tanh(\zeta))^n}{\cosh(\zeta)}|nn> \end{aligned} \quad (2)$$

where $\zeta > 0$ is the squeezing parameter and $|nn> \equiv |n_a> \otimes |n_b>$. In analogy with the harmonic oscillator it proofs convenient to introduce "position and momentum operators" through the relation $(a = (\mathbf{q} + i\mathbf{p})/\sqrt{2})$ and similarly for the second channel. As usual, eigenstates of the number operator $N_a = a^\dagger a$ ($N_b = b^\dagger b$) are denoted by $|n>$. For a single mode light field, the authors in [2] introduce the pseudospin operators for photons:

$$\begin{aligned} s_z &= \sum_{n=0}^{\infty}[|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|] \\ s_- &= \sum_{n=0}^{\infty}|2n\rangle\langle 2n+1| \\ s_+ &= (s_-)^\dagger, \quad s_\pm = \frac{1}{2}(s_x \pm i s_y) \end{aligned} \quad (3)$$

For an arbitrary unit vector $\mathbf{u}$ defined by the polar (θ_a) and azimuthal (ϕ_a) angles, the projection of the spin operator on the direction of $\mathbf{u}$ is given by:

$$\mathbf{u}\cdot\hat{s} = s_z\cos\theta_u + \sin\theta_u(e^{i\varphi_u}s_- + e^{-i\varphi_u}s_+)$$

Since the commutation relation for the operators s_z, s_- and s_+ are identical to those of the spin 1/2 system and given that for arbitrary unit vector $\mathbf{u}$, $(\mathbf{u}\cdot\hat{s})^2 = 1$ the authors in [2] conclude that there exist a perfect analogy between CV and the usual spin 1/2 systems. An alternative approach [10] based of the eigenstate $|q>$ of the position operator is available. In this case the parity operator is introduced:

$$\mathbf{\Pi}_z = \int_0^\infty \mathrm{dq}(|\varepsilon\rangle\langle\epsilon| - |\mathcal{O}\rangle\langle\mathcal{O}|) \quad (4)$$

while the x and y components are chosen as:

$$\mathbf{\Pi}_x = \int_0^\infty \mathrm{dq}(|\epsilon\rangle\langle\mathcal{O}| + |\mathcal{O}\rangle\langle\varepsilon|) \quad (5)$$

$$\mathbf{\Pi}_y = i\int_0^\infty \mathrm{dq}(|\mathcal{O}\rangle\langle\epsilon| - |\varepsilon\rangle\langle\mathcal{O}|) \quad (6)$$

where the even and odd eigenstates are related to the position eigenkets by the relations:

$$\begin{aligned} |\mathcal{E}> &= \frac{1}{\sqrt{2}}(|q> + |-q>) \\ |\mathcal{O}> &= \frac{1}{\sqrt{2}}(|q> - |-q>). \end{aligned} \quad (7)$$

The components of $\mathbf{\Pi}$ also satisfy the $SU(2)$ algebra.

For the CV case, the Bell operator is obtained by replacing in (1) the spin 1/2 operators either by the pseudospin (3) or by the Π operator in (4,5). The inequality is expressed in terms of the matrix elements of the Bell operator between TMSV states $|\zeta>$.

The Bell operator (1) involves four unit vectors. The following values for the angles are chosen: $\phi_u = \phi_v = \phi'_u = \phi'_v = 0$, so that the matrix element reduces to:

$$<\zeta|\mathcal{B}|\zeta> = \cos(\theta_u)\cos(\theta_v)I(\zeta) + \sin(\theta_u)\sin(\theta_v)F(\zeta) \quad (8)$$

where

$$\begin{aligned} I(\zeta) &= <\zeta|s_z^1 \otimes s_z^2|\zeta>, \\ F(\zeta) &= <\zeta|s_x^1 \otimes s_x^2|\zeta> \end{aligned} \quad (9)$$

If we further choose $\theta_u = 0$, $\theta'_u = \pi/2$, $\theta_v = -\theta'_v$, we obtain [2, 10]:

$$<\zeta|\mathcal{B}|\zeta> = 2\cos(\theta_v)I(\zeta) + 2\sin(\theta_v)F(\zeta) \quad (10)$$

Maximal violation of the inequality is obtained for $\tan(\theta_v) = F(\zeta)/I(\zeta)$ and amounts to:

$$<\zeta|\mathcal{B}|\zeta> = 2\sqrt{I(\zeta)^2 + F(\zeta)^2} \quad (11)$$

For the TMSV state $|\zeta>$, $I(\zeta)=1$ whereas $F(\zeta)$ depends upon our choice for the x,y components of the pseudospin operators. Thus for example [2, 10]:

$$F(\zeta) = \tanh(2\zeta) \quad for\ (3), \tag{12}$$

$$F(\zeta) = \frac{2}{\pi}\arctan(\sinh(2\zeta)) \quad for\ (5). \tag{13}$$

Note that in both cases the Cirel'son bound is attained in the $\zeta \to \infty$ limit, although for all ζ, $\tanh(2\zeta) \leq \frac{2}{\pi}\arctan(\sinh(2\zeta))$.

III. The role of degeneracy

The analogy between the CV system and the spin 1/2 is appealing, however it is clear that care must be exerted since, in general we do not expect both systems to have similar properties . For example, the number of degrees of freedom involved in both systems are not equal. An important difference between the conventional spin 1/2 and the pseudospin operators introduced in [2] is the degeneracy. For the spin 1/2 there is a unique $|+>$ state such that $s_z|+>=\frac{\hbar}{2}|+>$ while for the pseudospin operator (3) all the states of the type $|2n_0+1>$ for $n_0=0,1,2,...\infty$ are eigenstates of the parity-spin with eigenvalue one. Thus, there is an infinite degeneracy. Similarly for spin 1/2 there is only one state such that $s_+|+>=0$ while for pseudospin all the states $|2n_0+1>$ with $n_0=0,1,2,...,\infty$ are annihilated by s_+. Similar results hold when eigenstates $|q>$ of the position operator are considered since in this case any symmetric (or antisymmetric) $|q_0>$ state are eigenstates of the parity operator.

When considering the violation of the Bell-CHSH inequality for squeezed states, an intriguing possibility is the alternative of redefining the operator. Indeed, instead of considering the matrix element between TMSV states, we can consider expectation values respect to the vacuum:

$$<\zeta|\mathcal{B}|\zeta>=<00|\tilde{\mathcal{B}}|00>, \tag{14}$$

where obviously

$$\tilde{\mathcal{B}} = S(\zeta)^\dagger \mathcal{B} S(\zeta) \tag{15}$$

Thus, instead of entangled states we have an entangled observable [9]. In this example in particular, the transformation $S(\zeta)$ is unitary and the non triviality of its action stems from the fact that it involves operators acting on different Hilbert spaces.

Thus, in order to evaluate the violation of the inequality, we first calculate the "entangled Bell CHSH" operator (15) and then take its vacuum expectation value. Since such an evaluation may become involved, we simplify the calculation considering different approximations. To this end notice that the analogy between the CV and the spin 1/2 does not require to keep an infinite number of terms in the definition of the pseudospin operators in (3). Thus for example, restraining the sum to the $N=0$ term, the resulting operators have the same properties in the corresponding subspace, as the full pseudospin operator. We can work with those limited operators so as to figure out the answer to the questions risen above.

Below we quote the expressions for the tensor product of the pseudospin components entering in $\tilde{\mathcal{B}(\zeta)}$ (see (15, 8, 9)) when the sum in (3)is limited to the first term:

$$\begin{aligned} &\cosh(\zeta)^2 s_z^{(1)}\otimes s_z^{(2)} \to e^{Kab} s_z^{(1)}\otimes s_z^{(2)} e^{Ka^\dagger b^\dagger} \\ =&\ (1+K^2)|00><00| - |01><01| - |10><10| \\ +&\ |11><11| + K(|11><00| + |00><11|) \end{aligned} \tag{16}$$

$$\begin{aligned} &\cosh(\zeta)^2 s_x^{(1)}\otimes s_x^{(2)} \to e^{Kab} s_x^{(1)}\otimes s_x^{(2)} e^{Ka^\dagger b^\dagger} \\ =&\ |00><11| - |01><10| - |10><01| \\ +&\ |11><00| + 2K|00><00|. \end{aligned} \tag{17}$$

It should be clear that in the case we are considering there is no degeneration. Since

$$I(\zeta)_{N=0} = \frac{1+K^2}{(\cosh\zeta)^2}, \qquad F(\zeta)_{N=0} = \frac{2K}{(\cosh\zeta)^2} \tag{18}$$

the Bell-CHSH inequality yields the following result:

$$<00|\tilde{\mathcal{B}(\zeta)}|00>_{N=0}= \frac{1+6K^2+K^4}{(\cosh\zeta)^4} \tag{19}$$

So far we know the vacuum expectation value of the Bell operator when there is no degeneracy (18,19), other cases may be treated along similar lines *i.e.* increasing the degree of degeneration or equivalently increasing number of terms one considers in the definition of the pseudospin operator [11]. Although the evaluation of the tensor product $e^{Kab}s_i^{(1)}\otimes s_i^{(2)}e^{Ka^\dagger b^\dagger}$ very rapidly becomes cumbersome, the calculation is simplified remembering that we only need the vacuum expectation value of these operators. The expansion of the operator (e^{Kab}) involves equal powers of the operators a and b, so that only terms of equal occupation number will survive the matrix elements, so we write for the tensor products:

$$\begin{aligned} s_z^{(1)}\otimes s_z^{(2)} &= \sum_{n=0}^{i} |n,n><n,n| + h.c. \\ s_x^{(1)}\otimes s_x^{(2)} &= \sum_{n=0}^{i} |2n,2n><2n+1,2n+1| + h.c. \end{aligned} \tag{20}$$

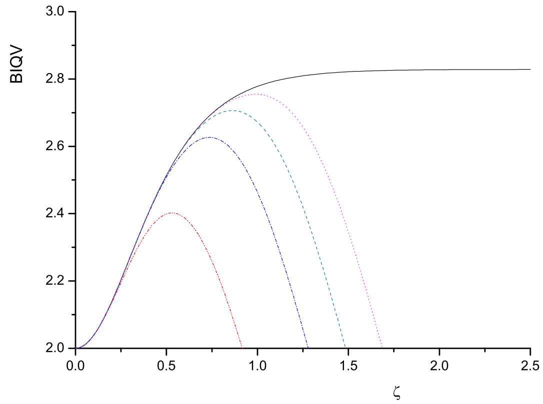

FIG. 1: Bell inequality violation (BIQV) as a function of the squeezing parameter ζ. The continuous line corresponds to ∞ degeneration. The other lines correspond to degeneracy equal to 3,2,1 and 0, in that order, from top to bottom.

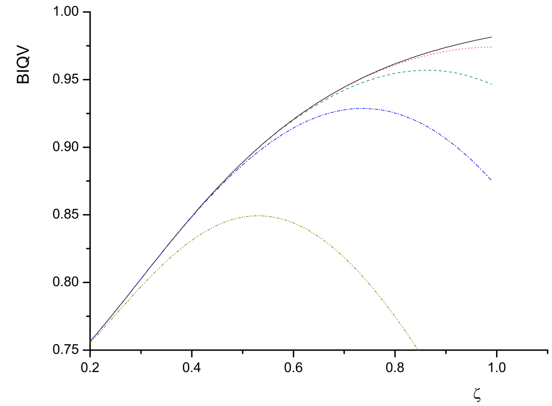

FIG. 2: Ratio of the Bell inequality violation (BIQV)to the maximal possible value ($2\sqrt{2}$) as a function of the squeezing parameter ζ. The continuous line corresponds to ∞ degeneration. The other lines correspond to degeneracy equal to 3,2,1 and 0, in that order, from top to bottom.

the upper limit indicates that we can stop the sum at the $i-th$ term, in whose case we will have degeneracy equal to i, $i=0$ corresponding to no degeneration. The action of the operators e^{Kab} to the left and $e^{Ka^\dagger b^\dagger}$ to the right is readily calculated by expanding them in power series, thus we obtain

$$\begin{aligned} e^{Kab}s_z^{(1)} &\otimes s_z^{(2)}e^{Ka^\dagger b^\dagger} = \sum_{n=0}^{i}(1+K^2)K^{4n} < 00| + T_1 \\ e^{Kab}s_i^{(1)} &\otimes s_i^{(2)}e^{Ka^\dagger b^\dagger} = 2\sum_{n=0}^{i}K^{4n+1}|00 >< 00| + T_2. \end{aligned} \tag{21}$$

where T_1 and T_2 stand for other terms whose vacuum expectation value vanishes. This is to be compared to the conventional calculation, where the $|\zeta>$ state is explicitly introduced:

$$\begin{aligned} I(\zeta) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{K^{n+m}}{\cosh(\zeta)^2} < mm|s_x^{(1)}\otimes s_x^{(2)}|nn > \\ &= 2\sum_{j=0}^{\infty}\frac{K^{4j+1}}{\cosh(\zeta)^2} = \tanh(2\zeta) \end{aligned} \tag{22}$$

$$\begin{aligned} F(\zeta) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{K^{n+m}}{\cosh(\zeta)^2} < mm|s_z^{(1)}\otimes s_z^{(2)}|nn > \\ &= \sum_{j=0}^{\infty}\frac{K^{2j}}{\cosh(\zeta)^2} = 1. \end{aligned} \tag{23}$$

(21) and (22,23) lead to the same result when an infinite number of terms are included. The advantage of (21) is that it permits to quantify the role of degeneracy in the violation of Bell's inequality. In Fig(1) we show the vacuum expectation value of the Bell operator, as a function of ζ, for different levels of degeneration . In particular, the results for infinite degeneration ($i=\infty$), obtained by Chen *et al* [2] and the result when no degeneration ($i=0$) is present (19) are shown. Notice that the behavior for small values of ζ is as interesting as the $\zeta\to\infty$ region. Indeed, for the two mode squeezed vacuum states (TMSV) and for $\zeta\to 0$, we know the violation tend to vanish. In such a limit all the approximations (different level of degeneration) coincide with the exact result. However, for $1/2<\zeta<1$ we observe an important violation of the Bell-CHSH inequality. An alternative way to present this information is by plotting (see Fig(2)) the ratio of the expectation value of the Bell operator for different level of degeneration to the maximal possible value ($2\sqrt{2}$). Clearly the larger the degeneration the closer (and always below) the vacuum expectation value to the behavior obtained for infinite degeneracy.

Maximal violation of Bell‘s inequalities is predicted only for infinite squeezing, which is not experimentally easy to realize. In this respect note that for $\zeta\approx 1/2$ and no degeneration (only the $|00>, |01>, |10>$ and $|11>$ states are involved), the violation of Bell‘s inequality may be as large as 85% of the Cirel'son bound. This may be relevant when considering the robustness against photon losses of continuous variable systems.

IV. Summary

In summary, in this paper we:

- Qualified the statement regarding the "perfect analogy" between a system with continuous variables and a spin 1/2, by remarking the existence of an infinite degeneration associated to the pseudospin operators introduced by authors in [2].
- Worked out an example that permit us to introduce the concept of entangled operator and also allow us to quantify the degree of violation of Bell's inequalities as a function of the degeneration and the squeeze parameter.
- Found a large ($> 85\%$) violation of the Bell-CHSH inequality in the non-degenerate case. For infinite degeneracy maximal violation is recovered.

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[1] K. Banaziek and K. Wodkiewicz, Phys. Rev. A 58 (1998) 4345; Phys. Rev. Letts. 82 (1999) 2009.

[2] Zeng-Bing Chen, Jian-Wei Pan, Guang Hou and Yong-De Zhang, Phys. Rev. Letts. 88 (2002) 04046; L. Mysta, Jr., R. Filip and J. Fiurasek, Phys. Rev. A65 (2002) 062315. See also C. Brukner, M.S. Kim, Jian-Wei Pan and A. Zeilinger, Phys. Rev A68 (2003) 062105.

[3] H. Jeong, W. Son, M.S. Kim, D. Ahn and C. Bruckner, Phys. Rev. A67 (2003) 012106.

[4] N. Gisin and A. Peres, Phys. Letts. A 162 (1992) 15.

[5] A detailed account of the efforts to describe non-locality and entanglement both for discrete and continuous variables can be found in the review article by S.L. Braunstein, and P. Loock, Reviews of Modern Physics 77 (2005) 513.

[6] J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press (1987) 139.

[7] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47 (1935) 777.

[8] S. L. Braunstein, A. Mann and M. Revsen, Phys. Rev. Letts. 68 (1992) 3259.

[9] A.S. Trushechkim and I.V. Volovich, ArXiv: quant-ph/0504156.

[10] G. Gour, F.C. Khana, A. Mann, M. Revzen, Phys. Letts. A 324 (2004) 415.

[11] The number of terms to be included for each level of degeneration is fixed so that the commutation relations are fulfilled. Due to the form of s_+ ($|2n_0+1><2n_0|$), the minimal elements necessary to define a spin operator are two. Correspondingly the number of terms required for s_z are two ($|2n_0+1><2n_0+1| - |2n_0><2n_0|$). Thus, every time we increase in one unit the level of degeneration we must add one term to $s_\pm$ and two to s_z